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Existence of equivariant h-cobordisms with given Whitehead torsions

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In the present paper, we prove an equivariant version of the existence theorem of an h-cobordism with a given torsion. Let G be a Lie group acting properly and smoothly on smooth manifolds W and M which is a submanifold of W, and we suppose that M/G and W/G are compact. We note that G, M and W are possibly non-compact.

<u>Theorem 1 (G-Existence Theorem).</u> Let G be a Lie group and M be a G-manifold as above. Suppose that M satisfies the conditions (1) and (2).

(1) (Codimension ≥ 3 condition).

If $M^{H_i}{}_{\alpha} \supset M^{H_j}{}_{\beta}$, then

$\dim M^{H_i}{}_{\alpha} - \dim (M^{H_i}{}_{\alpha} \cap G \cdot M^{H_j}{}_{\beta}) \geq 3$

for any pair of components $M^{H_i}{}_{\alpha}$ and $M^{H_j}{}_{\beta}$.

(2) (Higher dimension condition).

$\dim M_{i\,\alpha} / W_{\alpha} H_i \geq 5$ for any components $M^{H_i}{}_{\alpha}$.

Then for each $\sigma \in Wh_G(M)$, there exists a G-h-cobordism $(W;M,M')$ such that $\tau(W,M) = \sigma$.

The notions appeared in above theorem will be defined below in §2 and §3.

In [5] S. Illman introduced a general equivariant simple homotopy theory when G is a compact Lie group. Furthermore he defined the equivariant Whitehead group $Wh_G(X)$ of a finite G-CW complex X and the equivariant Whitehead torsion $\tau(f) \in Wh_G(X)$ of a G-homotopy equivalence $f : X \to Y$ between finite G-CW complexes. The group $Wh_G(X)$ is defined in a geometric way in analogy with the geometric definition of the ordinary Whitehead group. In [4] H. Hauschild gave an algebraic description of $Wh_G(X)$. To prove the existence theorem we take advantage of this method that it gives the chain complexes from which the torsion invariants are to be computed, see § 4. By the analogous method, S. Illman proved that equivariant Whitehead torsion is a combinatorial invariant in [6]. This is impotant to know since equivariant Whitehead torsion is not a topological invariant.

In [1], Araki and kawakubo proved an equivariant version of the s-cobordism theorem when G is a compact Lie group and M is a compact G-manifold.
Unfortunately the G-s-cobordism theorem does not hold in general, so they need to add some assumptions for the theorem. These results hold under our situation, and we can replace these assumptions with the conditions (1) and (2) above in the Theorem 1. It follows from the G-s-cobordism theorem and Theorem 1 that the G-h-cobordism is unique for a given Whitehead torsion. So we can classify G-h-cobordisms in terms of the equivariant Whitehead torsions.

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§ 2. Preliminaries

We denote by G_x the isotropy group of G at $x \in M$, i.e., $G_x = \{ g \in G \mid gx = x \}$. For any isotropy group G_x, we denote by (G_x) the conjugacy class of G_x in G, and we call it type of x. Since M/G is compact, there is a only finite number of isotropy types, as we now prove:

We prove by induction on the dimention of M. Suppose that $\dim M = 0$, there is a only finite number of isotropy types, since a compact 0-dimensional manifold M/G consists of finite number of points. Next we assume that it holds for the case where the number of the dimension of M is less than k. Let M be an arbitrary smooth G-manifold with dimension k such that M/G is compact. It follows from the slice theorem that an open tubular neighborhood of any orbit space is G-diffeomorphic to $G \times_{G_x} S_x$, where S_x is a slice of x. Then we have an open covering of M ; $\{ G \times_{G_x} S_x \}_{x \in M}$, and an open covering of M/G; $\{(G \times_{G_x} S_x)/G \}_{x \in M}$. Since M/G is compact we can choose a finite number of $x \in M$ such that $\{(G \times_{G_x} S_x)/G \}$ is also an open covering of M/G.

So it is enough to show that there is an only finite number of isotropy types appearing in $G\times_{G_x}S_x$. We now denote

$$\nu = G\times_{G_x}S_x.$$

The isotropy group of G at $(g,v)\in\nu$ is of form as follows,

$$G_{(g,v)} = g\cdot(G_x)_v\cdot g^{-1}.$$

Since G_x acts S_x linearly, we have that

$$(G_x)_v = (G_x)_{tv},$$

for any $t(\neq 0)\in\mathbb{R}$, i. e.,

$$G_{(g,v)} = G_{(g,tv)},$$

for any $t(\neq 0)\in\mathbb{R}$. As is well known, there is a G-invariant Riemannian metric on ν, see [8]. Let $S(\nu)$ be a unit sphere bundle. Obviously we have that

$$\{\text{ type appearing in } \nu\ \}$$
$$= \{\text{ type appearing in } S(\nu)\ \}\cup\{\ (G_x)\ \}.$$

$S(\nu)$ is a k-1 dimensional smooth G-manifold. Thus we have shown that there is a finite number of isotropy types appearing in $S(\nu)$. It follows the result.

Then we denote

$$\{(G_x)\ |\ x\in M\ \} = \{(H_1),(H_2),\cdots,(H_k)\}.$$

It is possible to arrange $\{(H_i)\}$ in such an order that $(H_i)\supset(H_j)$ implies $i\leq j$, where $(H_i)\supset(H_j)$ means that a conjugate of H_j is contained in H_i.

Next we recall the definition of so called the Kawakubo filtration of (G,M), $M = M_1\supset M_2\supset\cdots\supset M_k$ in [2], which consists of G-manifolds with corners such that

$$\{(G_x) \mid x \in M_i\} = \{(H_i), (H_{i+1}), \cdots, (H_k)\}$$

as follows. We may identify the equivariant normal bundle ν_1 of $M^{(H_1)}$ in M_1 with an open tubular neighborhood of $M^{(H_1)}$ in M_1 and impose a G-invariant Riemannian metric on ν_1, see (6). Concerning the metric on ν_1, we set

$$M_2 = M - \mathring{\nu}_1(1),$$

where $\mathring{\nu}_1(\varepsilon)$ stands for the open disk bundle of radius ε in ν_1. Note that

$$\{(G_x) \mid x \in M_2\} = \{(H_2), (H_3), \cdots, (H_k)\}.$$

Suppose that we get a filtration $M = M_1 \supset M_2 \supset \cdots \supset M_i$ of M such that

$$\{(G_x) \mid x \in M_i\} = \{(H_i), (H_{i+1}), \cdots, (H_k)\}.$$

We may identify the equivariant normal bundle ν_i of $M_i{}^{(H_i)}$ in M_i with an open tubular neighborhood of $M_i{}^{(H_i)}$ in M_i and impose a G-invariant Riemannian metric on ν_i. Concerning the metric on ν_i, we set

$$M_{i+1} = M_i - \mathring{\nu}_i(1).$$

Note that

$$\{(G_x) \mid x \in M_{i+1}\} = \{(H_{i+1}), (H_{i+2}), \cdots (H_k)\}.$$

This completes the inductive construction.

Putting $X_i = G\backslash M_i$, we have a filtration $X = X_1 \supset X_2 \supset \cdots \supset X_k$ of X.

Let H_i be an isotropy group appearing in M. We denote

$$M^{<H_i>} = \{ x \in M \mid G_x = H_i \}$$

$$M^{(H_i)} = \{ x \in M \mid (G_x) = (H_i)\} = G \cdot M^{<H_i>}$$

$$M^{H_i} = \{ x \in M \mid hx = x \text{ for any } h \in H_i \}.$$

Let $M^{H_i} = \coprod_\lambda M^{H_i}{}_\lambda$ be the decompositions of M^{H_i} into connected components. We denote by WH_i the quotient group of the normalizer of H_i in G by H_i. The WH_i -action on M^{H_i} induces the WH_i-action on the set of connected components of M^{H_i}. Taking WH_i orbits of the induced action, we get a decomposition

$$M^{H_i} = \coprod_\alpha WH_i \cdot M^{H_i}{}_\alpha$$

as a topological sum of WH_i-subspaces, where $M^{H_i}{}_\alpha$'s are connected components of M^{H_i}. We denote

$$W_\alpha H_i = \{w \in WH_i \mid w \cdot M^{H_i}{}_\alpha \subset M^{H_i}{}_\alpha\}$$

which is a closed subgroup of WH_i. Then we put

$$M_{i\,\alpha} = M_i \cap M^{<H_i>}{}_\alpha,$$

$$X_{i\,\alpha} = X_i \cap X^{(H_i)}{}_\alpha, \quad \text{where} \quad X^{(H_i)}{}_\alpha = M^{(H_i)}{}_\alpha / G.$$

It is easy to see that

$$X^{(H_i)}{}_\alpha = M^{<H_i>}{}_\alpha / W_\alpha H_i,$$

$$X_{i\,\alpha} = M_{i\,\alpha} / W_\alpha H_i.$$

We now replace M by W, and consider two conditions.

(1)' (Codimension ≥ 3 condition).

If $W^{H_i}{}_\alpha \supset W^{H_j}{}_\beta$, then

$\dim W^{H_i}{}_\alpha - \dim (W^{H_i}{}_\alpha \cap G \cdot W^{H_j}{}_\beta) \geq 3$

for any pair of components $W^{H_i}{}_\alpha$ and $W^{H_j}{}_\beta$.

(2)' (Higher dimension condition).

$\dim W_{i\,\alpha} / W_\alpha H_i \geq 6$ for any components $W^{H_i}{}_\alpha$.

Note that H_i is a maximal isotropy group appearing in W_i. If W satisfies the conditions (1)' and (2)', the G-s-cobordism theorem holds. Furthermore an equivariant version of the s-cobordism theorem holds under our situations that G is a Lie group acting properly and smoothly on smooth manifolds M and W, and that M/G and W/G are compact.

$(W;M,M')$ is called a smooth G-h-cobordism, if W is a G-manifold with boundary $\partial W = M \amalg M'$ (disjoint union) and the inclusion maps

$$i: M \to W \quad \text{and} \quad i': M' \to W$$

are G-homotopy equivalences. Then we consider other conditions.

(1) (Codimension ≥ 3 condition).

If $M^{H_i}{}_{\alpha} \supset M^{H_j}{}_{\beta}$, then

$\dim M^{H_i}{}_{\alpha} - \dim (M^{H_i}{}_{\alpha} \cap G \cdot M^{H_j}{}_{\beta}) \geq 3$

for any pair of components $M^{H_i}{}_{\alpha}$ and $M^{H_j}{}_{\beta}$.

(2) (Higher dimension condition).

$\dim M_{i\,\alpha} / W_{\alpha} H_i \geq 5$ for any components $M^{H_i}{}_{\alpha}$.

It should be noted that a G-h-cobordism $(W;M,M')$ satisfies the conditions (1)' and (2)' if and only if it satisfies the conditions (1) and (2).

§3. Equivariant Whitehead torsions

In this section we first define the equivariant Whitehead group $Wh_G(M)$ for a smooth G-manifold M and try to decompose $Wh_G(M)$, refer to [3].

For a compact Lie subgroup H of G, $(G/H) \times D^n$ is a G-space together with a proper G-action. $(G/H) \times D^n$ is called an n-G-cell, and (H) is called (isotropy) type of the n-G-cell $(G/H\times)D^n$. Here D^n

is a unit n-disk of $\mathbb{R}^n$, and G acts D^n trivialy. By a finite relative G-CW complex (V,M), we shall mean a G-space together with a proper G-action such that V is obtained from a smooth G-manifold M by attaching a finite number of G-cells. We now consider the set,

$$A_G(M) = \{(V,M) \mid (V,M) \text{ is a finite relative G-CW complex, and } M \text{ is a G-deformation retract of } V\}.$$

Let (V_1,M) and (V_2,M) be elements of $A_G(M)$. If there is a formal G-deformation from V_1 to V_2 we write $V_1 \underset{G}{\curvearrowright} V_2$. This is clearly an equivalence relation and we let $\tau(V,M)$ denote the equivalence class of (V,M). An addition of equivalence classes is defined by setting

$$\tau(V_1,M) + \tau(V_2,M) = \tau(V_1 \cup_M V_2,\ M)$$

where $V_1 \cup_M V_2$ is the disjoint union of V_1 and V_2 identified by the identity map on M.

The equivariant Whitehead group for a smooth G-manifold M is defined to be the set of equivalence classes with the given addition and is denoted $Wh_G(M)$;

$$Wh_G(M) = A_G(M)/\sim\ ,$$

and a element $\tau(V,M)$ of $Wh_G(M)$ is called the Whitehead G-torsion of (V,M).

If $f : M_1 \to M_2$ is a G-map, we define

$$\begin{array}{rccc} f_\# : & Wh_G(M_1) & \to & Wh_G(M_2) \\ & \Psi & & \Psi \\ & \tau(V,M_1) & \to & \tau(V \cup_f M_2, M_2). \end{array}$$

It is known that $f_\# = g_\#$ if $f,\ g : M_1 \to M_2$ are G-homotopic, refer to (3). Let $r : V \to M$ be the G-retraction and M_r be the mapping cylinder of r. We put

$$\widetilde{M}_r = M_r / \sim ,$$

where $\sim$ means an equivalence relation that $M \times I$ and M identified by the projection map $p : M \times I \to M$.
Then we have

$$-\tau(V, M_1) = r_{\#}\ \tau(\widetilde{M}_r, V),$$

refer to (3). So $Wh_G(M)$ is an abelian group.

Now we review an algebraic decomposition of $Wh_G(M)$. We have a Lie group $\Gamma_{i\alpha}$ for each $W^{H_i}{}_\alpha$, satisfying the following short exact sequence;

$$1 \to \pi_1(M^{H_i}{}_\alpha) \to \Gamma_{i\alpha} \to W_\alpha H_i \to 1.$$

Then we have that

$$
\begin{aligned}
Wh_G(M) &\cong \coprod_{(H_i)} Wh_G(M, (H_i)) \\
&\cong \coprod_{(H_i)} Wh_{WH_i}(M^{H_i}, \{e\}) && \text{(see (4))} \\
&\cong \coprod_{(H_i),\ \alpha} Wh_{WH_i}(WH_i \cdot M^{H_i}{}_\alpha, \{e\}) \\
&\cong \coprod_{(H_i),\ \alpha} Wh_{W_\alpha H_i}(M^{H_i}{}_\alpha, \{e\}) \\
&\cong \coprod_{(H_i),\ \alpha} Wh\ \Gamma_{i\alpha}(\widetilde{M}^{H_i}{}_\alpha, \{e\}) \\
&\cong \coprod_{(H_i),\ \alpha} Wh_{alg}(\pi_0(\Gamma_{i\alpha})), && \text{(see (1))}
\end{aligned}
$$

where

$$Wh_G(M, (H_i)) = \{\tau(V, M) \in Wh_G(M) \mid (G_x) = (H_i) \text{ for any } x \in V - M\}.$$

§4. Proof of G-Existence theorem

At first we will show that

$$\coprod_{(H_i),\,\alpha} Wh_{alg}(\pi_0(\Gamma_{i\alpha})) = \coprod_{(H_i),\,\alpha} Wh_{alg}(\pi_1(X_{i\alpha})).$$

<u>Lemma 2</u>. If M satisfies the codimension ≥ 3 condition (1) above, then there is a natural isomorphism,

$$\partial : \pi_1(X_{i\alpha}) \to \pi_0(\Gamma_{i\alpha}).$$

<u>Proof</u>. It follows from the codimension ≥ 3 condition (1), and definition of $M_{i\alpha}$ that

$$\pi_1(M^{H_i}{}_{\alpha}) = \pi_1(M^{<H_i>}{}_{\alpha}) = \pi_1(M_{i\alpha}).$$

Since $\Gamma_{i\alpha}$ acts freely on the universal covering space $\widetilde{M}_{i\alpha}$ of $M_{i\alpha}$ and since $W_\alpha H_i$ acts freely on $M_{i\alpha}$, we have a fibration

$$\begin{array}{ccccc}
\widetilde{M}^{H_i}{}_{\alpha} & \supset & \widetilde{M}_{i\alpha} & \circlearrowleft & \Gamma_{i\alpha} \\
 & & \downarrow & & \\
M^{H}{}_{\alpha} & \supset & M_{i\alpha} & \circlearrowleft & W_\alpha H_i \\
 & & \downarrow & & \\
 & & X_{i\alpha} & &
\end{array}$$

From the homotopy exact sequence

$$\begin{array}{ccccccccc}
\to \pi_1(\widetilde{M}_{i\alpha}) & \to & \pi_1(\Gamma_{i\alpha}\backslash\widetilde{M}_{i\alpha}) & \to & \pi_0(\Gamma_{i\alpha}) & \to & \pi_0(\widetilde{M}_{i\alpha}) \to \\
\| & & \| & & & & \| \\
\{1\} & & \pi_1(X_{i\alpha}) & & & & \{0\}
\end{array}$$

follows the result. $\square$

Thus we can write that $\tau = \coprod_{(H_i),\alpha} \tau_{i\alpha}$ for $\tau_{i\alpha} \in Wh_{alg}(\pi_1(X_{i\alpha}))$.

Now, each $M_{i\alpha}$ is a principal $W_\alpha H_i$-bundle over $X_{i\alpha}$. Let V be a $W_\alpha H_i$-CW complex such that $\tau(V, M_{i\alpha}) \in Wh_{W_\alpha H_i}(M_{i\alpha}, \{e\})$. Note that a fixed point free formal $W_\alpha H_i$-deformation of the total space $(V, M_{i\alpha})$ induces a unique formal deformation of $(V/W_\alpha H_i,\ M_{i\alpha}/W_\alpha H_i) = (K, X_{i\alpha})$ and vice versa. It follows that the projection map $M_{i\alpha} \to X_{i\alpha}$ induces an isomorphism

$$\phi : Wh_{W_\alpha H_i}(M_{i\alpha}, \{e\}) \to Wh_{\{e\}}(X_{i\alpha}) \cong Wh_{alg}(\pi_1(X_{i\alpha})).$$

Then we have

<u>Lemma 3</u>. The inclusion map $\eta : M_{i\alpha} \to M^{H_i}{}_\alpha$ induces an isomorphism

$$\eta_* : Wh_{W_\alpha H_i}(M_{i\alpha}, \{e\}) \to Wh_{W_\alpha H_i}(M^{H_i}{}_\alpha, \{e\})$$

for any α and i, making the following diagram commute.

$$\begin{array}{ccc} Wh_{W_\alpha H_i}(M_{i\alpha}, \{e\}) & \xrightarrow{\eta_*} & Wh_{W_\alpha H_i}(M^{H_i}{}_\alpha, \{e\}) \\ \downarrow \phi & & \downarrow \phi \\ Wh_{alg}(\pi_1(X_{i\alpha})) & \xrightarrow{\partial_*} & Wh_{alg}(\pi_0(\Gamma_{i\alpha})) \end{array}$$

Here ∂_* is an isomorphism induced by the isomorphism ∂ obtained in Lemma 2 and ϕ is appeared in algebraic decomposition in (2).

<u>Proof</u>. For any $\tau(V, M_{i\alpha}) \in Wh_{W_\alpha H_i}(M_{i\alpha}, \{e\})$, the image $\phi(\tau(V, M_{i\alpha}))$ is nothing but the torsion $\tau(C)$

of the chain complex $C = \{C_j\}$;

$$C_j = H_j(\tilde{K}^j \cup \tilde{X}_{i\alpha},\ \tilde{K}^{j-1} \cup \tilde{X}_{i\alpha})$$

$$= H_j(\coprod_{k=1,\cdots,m} \pi_1(X_{i\alpha}) \times E^j{}_k,\ \coprod_{k=1,\cdots,m} \pi_1(X_{i\alpha}) \times \partial E^j{}_k)$$

$$= \mathbb{Z}((\pi_1(X_{i\alpha}))) \otimes \{e^j{}_1, e^j{}_2, \cdots, e^j{}_m\}$$

where $\tilde{K}$ is the universal covering of K, $\tilde{K}^j$ is the underlying topological space of the j-skeleton of $\tilde{K}$, E^j is a j-cell of $\tilde{K} - \tilde{X}_{i\alpha}$, and m is the number of j-cells which are contained in $K - X_{i\alpha}$.

On the other hand

$\eta_*(\tau(V, M_{i\alpha})) = \tau(V \cup M^{H_{i_\alpha}}, M^{H_{i_\alpha}}) \in Wh_{W_\alpha H_i}(M^{H_{i_\alpha}}, \{e\})$.

Hence $\psi \circ \eta_*(\tau(V, M_{i\alpha}))$ is the torsion $\tau(C')$ of the chain complex $C' = \{C'_j\}$;

$$C'_j = H_j(\tilde{V}^j \cup \tilde{M}^{H_{i_\alpha}},\ \tilde{V}^{j-1} \cup \tilde{M}^{H_{i_\alpha}})$$

$$= H_j(\coprod_{k=1,\cdots,m} \Gamma_{i\alpha} \times \tilde{E}^j{}_k,\ \coprod_{k=1,\cdots,m} \Gamma_{i\alpha} \times \partial \tilde{E}^j{}_k))$$

$$= \mathbb{Z}(\pi_0(\Gamma_{i\alpha})) \otimes \{\tilde{e}^j{}_1, \cdots, \tilde{e}^j{}_m\},$$

where V^j is the underlying topological space of the $W_\alpha H_i$-j-skeleton of V, $\tilde{E}^j$ is a j-cell of $\tilde{V}^j \cup \tilde{M}^{H_{i_\alpha}}$ which is a lift of E^j, and m is the number of $W_\alpha H_i$-j-cells which are contained in $V - M_{i\alpha}$, see [2].

It suffices to prove that $\partial_*\tau(C) = \tau(C')$. Put

$$C''_j = H_j(\tilde{V}^j \cup \tilde{M}_{i\alpha},\ \tilde{V}^{j-1} \cup \tilde{M}_{i\alpha})$$

$$= H_j(\coprod_{k=1,\cdots,m} \Gamma_{i\alpha} \times \tilde{E}^j{}_k,\ \coprod_{k=1,\cdots,m} \Gamma_{i\alpha} \times \partial \tilde{E}^j{}_k))$$

$$= \mathbb{Z}(\pi_0(\Gamma_{i\alpha})) \otimes \{\tilde{e}^j{}_1, \cdots, \tilde{e}^j{}_m\}.$$

Let $\Gamma_{i\alpha,0}$ be the component of $\Gamma_{i\alpha}$ including the unit element. Since $\tilde{V}/\Gamma_{i\alpha 0}$ is a covering space of K and $\pi_1(\tilde{V}/\Gamma_{i\alpha 0})$ is trivial, we may regard $\tilde{K}$ as the quotient space of $\tilde{V}$ by the action of $\Gamma_{i\alpha,0}$. Let $q : \tilde{V} \to \tilde{K}$ be the quotient map. Then we may regard q as a fiber preserving map between the $\Gamma_{i\alpha}$-bundle $\tilde{V}$ and the $\pi_1(X_{i\alpha})$-bundle $\tilde{K}$ which have the same base space K. The restriction of q to $\tilde{M}_{i\alpha}$ is a fiber preserving map between the sub $\Gamma_{i\alpha}$-bundle $\tilde{M}_{i\alpha}$ and the sub $\pi_1(X_{i\alpha})$-bundle $\tilde{X}_{i\alpha}$, so we have the following commutative diagram between two exact sequences.

$$\begin{array}{ccccccccc}
\to & \pi_1(\tilde{M}_{i\alpha}) & \to & \pi_1(X_{i\alpha}) & \xrightarrow{\partial} & \pi_0(\Gamma_{i\alpha}) & & \to \\
 & \downarrow & & \| & & \downarrow f & & \\
\to & \pi_1(\tilde{X}_{i\alpha}) & \to & \pi_1(X_{i\alpha}) & \underset{id}{\xrightarrow{\partial}} & \pi_0(\pi_1(X_{i\alpha})) & & \to
\end{array}$$

Thus $f: \pi_0(\Gamma_{i\alpha}) \to \pi_0(\pi_1(X_{i\alpha})) = \pi_1(X_{i\alpha})$ is the isomorphism ∂^{-1}. The action of $\pi_1(X_{i\alpha})$ on $\tilde{K}$ can be identified that of $\pi_0(\Gamma_{i\alpha})$ via the homomorphism

$$\Gamma_{i\alpha} \to \Gamma_{i\alpha,0} \backslash \Gamma_{i\alpha} = \pi_0(\Gamma_{i\alpha})$$

which is induced by q. So the quotient map q induces a $\mathbb{Z}(\pi_0(\Gamma_{i\alpha}))$-homomorphism $q_{\#} : C''_j \to C_j$, if we regard C_j as a $\mathbb{Z}(\pi_0(\Gamma_{i\alpha}))$-module via f. On the other hand we have an excision isomorphism

$i_{\#} : C''_j \to C'_j$ induced by the inclusion map

$\tilde{V} \cup \tilde{M}_{i\alpha} \to \tilde{V} \cup \tilde{M}^{H_{i_\alpha}}$, since

$$\{(\tilde{V}^j) \cup \tilde{M}_{i\alpha}\} - \{(\tilde{V}^{j-1}) \cup \tilde{M}_{i\alpha})\}$$

$$= \{(\tilde{V}^j) \cup \tilde{M}^{H_{i_\alpha}}\} - \{(\tilde{V}^{j-1}) \cup \tilde{M}^{H_{i_\alpha}}\}.$$

We may identify two $\mathbb{Z}(\pi_0(\Gamma_{i\alpha}))$-modules C'_j and C''_j by $i_{\#}$.

We now put $\tau(C) = (a_{lk}) \in Wh_{alg}(\pi_1(X_{i\alpha}))$ where $a_{lk} \in \mathbb{Z}(\pi_1(X_{i\alpha}))$. Let $\bar{f}$ and $\bar{\partial}$ be isomorphisms between the integral group rings induced by f and ∂, respectively. Then

$$\phi \cdot \eta_*(\tau(V, M_{i\alpha})) = (\bar{f}^{-1}(a_{lk}))$$

$$= (\bar{\partial}(a_{lk})) \in Wh_{alg}(\pi_0(\Gamma_{i\alpha})).$$

This completes the proof of Lemma 3. □

We now construct an (n+1)-dimensional smooth G-manifold W with $\tau(W, M) = \tau$, where $\dim M = n$. From the higher dimension condition (2), we have an h-cobordism $(Y_{i\alpha}; X_{i\alpha})$ with $\tau(Y_{i\alpha}, X_{i\alpha}) = \tau_{i\alpha}$. $Y_{i\alpha}$ is obtained from $X_{i\alpha} \times I$ by attaching handles of indices 2 and 3 to $X_{i\alpha} \times \{1\}$, see [8], where $I = [0, 1]$. Let

$$r_{i\alpha} : Y_{i\alpha} \to X_{i\alpha}$$

be a smooth retraction. We have an induced smooth bundle $r^*(G \cdot M_{i\alpha})$. By the projection

$$\pi_{i\alpha} : r^*(G \cdot M_{i\alpha}) \to G \cdot M_{i\alpha},$$

we have again an induced bundle $\pi_{i\alpha}{}^*(\nu_{i\alpha}(1/2))$, where $\nu_{i\alpha}(1/2)$ is a closed tubular neighborhood of $G \cdot M_{i\alpha}$. Note that $\pi_{i\alpha}{}^*(\nu_{i\alpha}(1/2))$ is an (n+1)-dimensional smooth G-manifold. Let

$$r_{i\alpha}' = r_{i\alpha} \mid X_{i\alpha} \times I$$

$$\pi_{i\alpha}' = \pi_{i\alpha} \mid r_{i\alpha}'^*(G \cdot M_{i\alpha}).$$

Then we have that

$$r_{i\alpha}{}^*(G \cdot M_{i\alpha}) \supset r_{i\alpha}'^*(G \cdot M_{i\alpha}) = G \cdot M_{i\alpha} \times I$$

$$\pi_{i\alpha}{}^*(\nu_{i\alpha}(1/2)) \supset \pi_{i\alpha}'^*(\nu_{i\alpha}(1/2))$$

$$= \nu_{i\alpha}(1/2) \times I$$

and a commutative diagram with fiber bundles in the vertical:

$$\begin{array}{ccccc}
\nu_{i\alpha}(1/2) & \leftarrow & \pi_{i\alpha}^{*}(\nu_{i\alpha}(1/2)) & \supset & \pi_{i\alpha}'^{*}(\nu_{i\alpha}(1/2)) \\
\downarrow & & \downarrow & & \\
G\cdot M_{i\alpha} & \overset{\pi_{i\alpha}}{\leftarrow} & r_{i\alpha}^{*}(G\cdot M_{i\alpha}) & \supset & r_{i\alpha}'^{*}(G\cdot M_{i\alpha}) \\
\downarrow & & \downarrow & & \\
X_{i\alpha} & \overset{r_{i\alpha}}{\leftarrow} & Y_{i\alpha} & &
\end{array}$$

By the definition of the Kawakubo filtration, we have

$$M = \bigcup_{(H_i),\,\alpha} \nu_{i\alpha}(1) = \coprod_{(H_i),\,\alpha} \nu_{i\alpha}(1) \;/\sim$$

$$M \times I = \bigcup_{(H_i),\,\alpha} \nu_{i\alpha}(1) \times I = \coprod_{(H_i),\,\alpha} \pi_{i\alpha}'^{*}(\nu_{i\alpha}(1)) /\sim \times I$$

and thus

$$W = \coprod_{(H_i),\,\alpha} \{\pi_{i\alpha}^{*}(\nu_{i\alpha}(1/2)) \cup \pi_{i\alpha}'^{*}(\nu_{i\alpha}(1))\} /\sim \times I$$

(see figure 1).

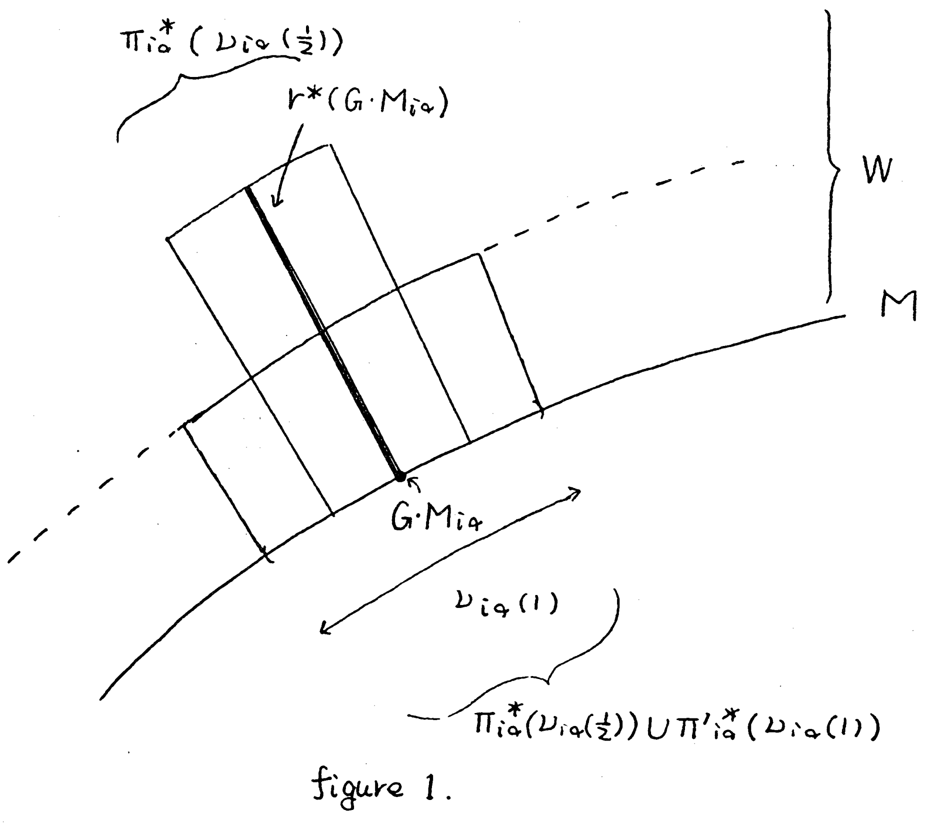

figure 1.

By smoothing the corners, we may assume W to be smooth.

Finally we will show that $\tau(W,M) = \tau$. We put $W' = \coprod_{(H_i),\,\alpha} (r_{i\alpha}{}^*(G\cdot M_{i\alpha}) \cup \pi_{i\alpha}{}'^*(\nu_{i\alpha}(1))) \,/\!\sim \times I.$

Since $\nu_{i\alpha}(1/2)$ collapses to $G\cdot M_{i\alpha}$ for all i and α, (W,M) collapses to (W',M), (compare figure 2).

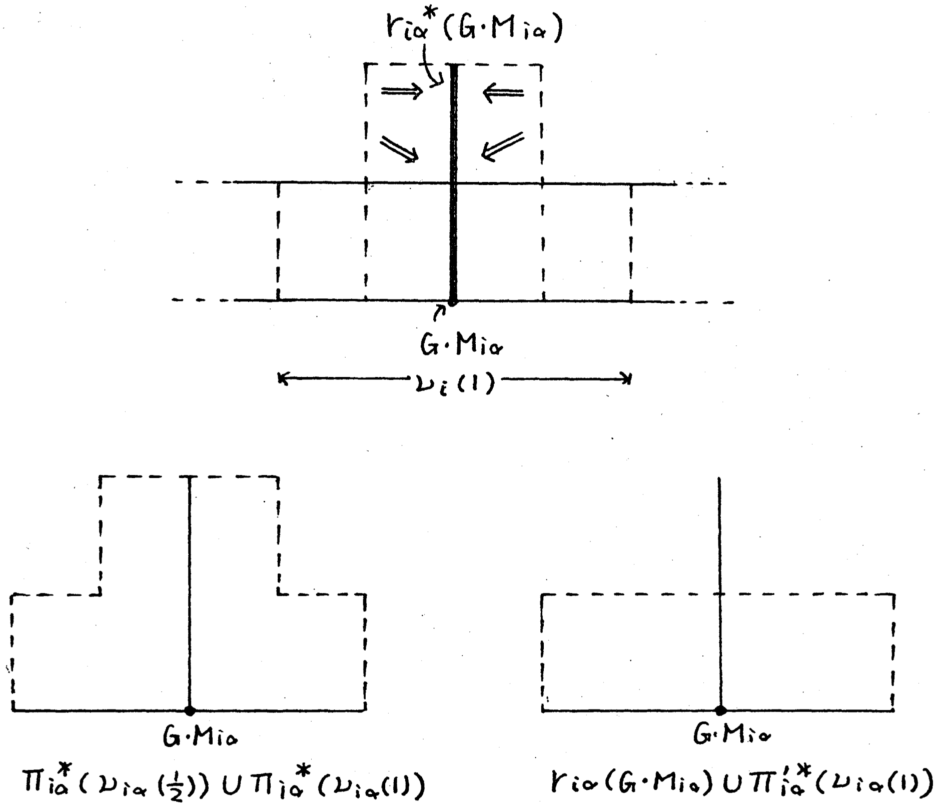

figure 2.

Thus we have

By Lemma 2, we get

$$\tau(W',M) = {}_{(H_i),\,\alpha}\, \tau_{i\alpha}.$$

This completes the proof of Theorem 1. □

In the same manner as in the non equivariant case, we can prove

<u>Theorem 4 (G-Uniqueness Theorem)</u>. Let $(W_1; M, M_1)$ and $(W_2; M, M_2)$ be two G-h-cobordisms which satisfy the conditions (1) and (2) above. If $\tau(W_1, M) = \tau(W_2, M)$, then we have a G-diffeomorphism

$$W_1 \cong W_2 \text{ rel } M.$$

So we can classify G-h-cobordisms in terms of the equivariant Whitehead torsions.

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